BASIC LOGIC AND QUANTUM ENTANGLEMENT

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Abstract

As it is well known, quantum entanglement is one of the most important features of quantum computing, as it leads to massive quantum parallelism, hence to exponential computational speed-up. In a sense, quantum entanglement is considered as an implicit property of quantum computation itself. But...can it be made explicit? In other words, is it possible to find the connective "entanglement" in a logical sequent calculus for the machine language? And also, is it possible to "teach" the quantum computer to "mimic" the EPR "paradox"? The answer is in the affirmative, if the logical sequent calculus is that of the weakest possible logic, namely Basic logic.

A weak logic has few structural rules. But in logic, a weak structure leaves more room for connectives (for example the connective "entanglement"). Furthermore, the absence in Basic logic of the two structural rules of contraction and weakening corresponds to the validity of the no-cloning and no-erase theorems, respectively, in quantum computing.
1. Introduction

Our purpose is to obtain an adequate sequent calculus [1] for quantum computation [2]. In particular, we look for logical connectives corresponding to the physical links existing among qubits in the quantum computer, and the associated inference rules. To this aim, we will exploit Basic logic [3] and its reflection principle between meta-language and object language. The sequent calculus we are looking for should be able to reproduce two main features of quantum computing namely quantum superposition and quantum entanglement. These two features taken together (the so-called quantum massive parallelism) are in fact very important as they lead to quantum computational speed-up [4].

A logical interpretation of quantum superposition is straightforward in Basic logic, and is given in terms of the additive connective & = "with" (and of its symmetric, ∨ = "or") both present in linear [5] and Basic logics.

In this paper, we also propose a logical interpretation for quantum entanglement. Entanglement is a strong quantum correlation, which has no classical analogous. Then, the logic having room for the connective "entanglement", will be selected as the most adequate logic for quantum mechanics, and in particular for quantum computing. Quantum entanglement is mathematically expressed by particular superposition of tensor products of basis states of two (or more) Hilbert spaces such that the resulting state is non-separable. For this reason, one can expect that the new logical connective, which should describe entanglement, will be both additive and multiplicative, and this is in fact the case. We introduce the connective @ = "entanglement" by solving its definitional equation, and we get the logical rules for @. It turn out that @ is a (right) connective given in terms of the (right) additive conjunction & and of the (right) multiplicative disjunction $\otimes$ = "par".

Then, we discuss the properties of @. In particular, we prove that @ is not idempotent, which is equivalent to formulate the "no self-referentiality" theorem in the meta-language.

Also, we show that, like all the connectives of Basic logic, @ has its symmetric, the (left) connective $\otimes$, given in terms of the (left) additive disjunction $\vee$ = "or", and the (left) multiplicative conjunction $\otimes$ = "times" (the symmetric of $\otimes$).

Moreover, we provide Basic logic of a new meta-rule, which we name EPR-rule as it is the logical counterpart of the so-called EPR "paradox" [6].

The conclusion of this paper is that Basic logic is the unique adequate logic for quantum computing, once the connective entanglement and the EPR-rule are included.

2. A brief review of basic logic

Basic logic [3] is the weakest possible logic (no structure, no free contexts) and was originally conceived [7] as the common platform for all other logics (linerar, intuitionistic, quantum, classical etc.) which can be considered as its "extensions". Basic logic has tree main properties:

i) Reflection: All the connectives of Basic logic satisfy the principle of reflection, that is, they are introduced by solving an equation (called definitional equation), which "reflects" meta-linguistic links between assertions into the object-language. There are only two metalinguistic links: "yields", "and". The metalinguistic "and", when is outside the sequent, is indicated by and; when inside the sequent, is indicated by a comma.

Object language $\rightarrow$ Reflection $\rightarrow$ (iff) Meta-language

connectives meta-linguistic links

 yields $-$

"and" (and outside the sequent)

(comma) inside the sequent

ii) Symmetry: All the connectives are divided into "left" and "right" connectives.

A left connective has formation rule acting on the left, and a reflection rule acting on the right. In Basic logic, every left connective has its symmetric, a right connective, which has a formation rule acting on the right, and a reflection rule acting on the left (and vice-versa).

Left connectives $\rightarrow$ Symmetry $\rightarrow$ Right connectives

$\vee$ = "or" (Additive disjunction) $\&$ = "with" (Additive conjunction)

$\otimes$ = "times" (Multiplicative conjunction) $\otimes$ = "par" (Multiplicative disjunction)

$\rightarrow$ (Contraposition) $\rightarrow$ (Implication)

iii) Visibility: There is a strict control on the contexts, that is, all active formulas are isolated from the contexts, and they are visible.

In Basic logic, the identity axiom $A \rightarrow A$, and the cut rule: $\Gamma, A \rightarrow \Delta$, and the cut rule: $\Gamma \rightarrow \Delta$ hold.

The cube of logics [3] [7] is a geometrical symmetry in the space of logics, which becomes apparent once one takes Basic logic as the fundamental one. As we said above, Basic logic is the weakest logic, and all the other logics can be...
considered as its extensions. All the logics, which have no structural rules (called substructural logics or resources logics) are the four vertices of one same face of the cube, considered as the basis. They have many connectives, less structure, and less degree of abstraction. On the upper face of the cube, we have all the structural logics (they have fewer connectives, more structure, and a higher degree of abstraction). See Fig.1.

**Fig.1 The cube of logics**

**Substructural logics:**
- **B** = Basic logic
- **BL** = Basic logic + context on the left
- **BR** = Basic logic + context on the right
- **BRL** = Linear logic

**Structural logics:**
- **BS** = Quantum logic
- **BSR** = Paraconsistent logic
- **BSL** = Intuitionistic logic
- **BSRL** = Classical logic

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3. **Reasons why Basic logic is the right logic for quantum computing**

Basic logic has the following features, which are essential to describe quantum computation in logical terms:

a) It is **non-distributive** (because of the absence, on both sides of the sequent, of active contexts), and this is of course a first necessary requirement for any logic aimed to describe a quantum mechanical system.

b) It is **substructural**, i.e., it has no structural rules like contraction: \( \frac{\Gamma, A \vdash \Delta}{\Gamma, A \vdash \Delta} \) (data can be copied) and weakening: \( \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \) (data can be deleted), accordingly with the no-cloning [8] theorem and the no-erase theorem [9] respectively, in quantum computing. The only structural rule, which holds in Basic logic, is the exchange rule:

\[
\text{exchL} \quad \frac{\Gamma, A, B, \Gamma \vdash \Delta}{\Gamma, B, A, \Gamma \vdash \Delta} \quad \text{exchR} \quad \frac{\Gamma \vdash \Delta, A, B, \Delta}{\Gamma \vdash \Delta, B, A, \Delta}
\]

Then, for example, standard quantum logic [10], although being non-distributive, is excluded as a possible candidate for quantum computing because it has structural rules. Linear logic is substructural, but has both left-side and right-side free contexts, then is excluded because of distributivity. (In particular, as we will see, for the connective \( \otimes \) ="entanglement", the distributive property does not hold, then Linear logic cannot accommodate \( \otimes \)).

c) It is **paraconsistent**: the non-contradiction principle is invalidated, and quantum superposition can be assumed.
4. The logical connective $\&$ for quantum superposition

The unit of quantum information is the qubit $|Q\rangle = a|0\rangle + b|1\rangle$, which is a linear combination of the basis states $|0\rangle$ and $|1\rangle$, with complex coefficients $a$ and $b$ called probability amplitudes, such that the probabilities sum up to one: $|a|^2 + |b|^2 = 1$. In logical terms, we will interpret the atomic proposition $A$ as bit $|1\rangle$, and its primitive negation $\bar{A}$ as bit $|0\rangle = NOT|1\rangle$, where NOT is the $2 \times 2$ off-diagonal matrix $\text{NOT} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The atomic assertion $\neg A$ will be interpreted as the quantum state $\neg A \equiv |A\rangle = b|1\rangle$, and its negation as $\neg \neg (A^\perp) \equiv |A^\perp\rangle = a\text{NOT}|1\rangle = a|0\rangle$. Notice that making the negation of the atomic assertion $\neg A$ is not the same as asserting the negation of the atomic proposition, in fact $(\neg A)^\perp \equiv \text{NOT}|A\rangle = b|0\rangle$. (In the case of basic logic, it can be shown [11] that, when the meta-language is that of quantum mechanics, truth-values can be semantically interpreted as probabilities, and sequents $\Gamma \vdash \Delta$ as scalar products).

In the meta-language, quantum superposition means that both propositions $A$ and $A^\perp$ are asserted, that is, on the right-hand side of the definitional equation, we will have: $\neg A$ and $\neg (A^\perp)$. On the left-hand side, we look for the connective $\&$="superposition", such that: $\neg A \& \neg (A^\perp)$.

This is the definitional equation of $\&$ [3]:

$$
\Gamma \vdash A \& B \quad \text{iff} \quad \Gamma \vdash \neg A \quad \text{and} \quad \Gamma \vdash \neg (A^\perp)
$$

In the particular case with $B = A^\perp$ and $\Gamma = \emptyset$. So that we can write the definitional equation for the connective "quantum superposition" as:

$$
\neg A \& A^\perp \quad \text{iff} \quad \neg A \quad \text{and} \quad \neg (A^\perp)
$$

(1)

Then, of course, the rules of the connective "quantum superposition" are the same rules of $\&$ [3], with $B = A^\perp$ and $\Gamma = \emptyset$.

$$
\& - \text{form} \quad \Gamma \vdash \neg A \quad \Gamma \vdash \neg (A^\perp) \quad \Gamma \vdash A \& B
$$

This is obtained from the RHS to the LHS of the definitional equation (1).

$$
\& - \text{implicit refl} \quad \Gamma \vdash A \& A^\perp \quad \Gamma \vdash A \quad \Gamma \vdash \neg (A^\perp)
$$

This is obtained from the LHS to the RHS of the definitional equation (1).

By trivializing the $\&$-implicit reflection, i.e., putting $\Gamma = A \& A^\perp$, we get the two $\&$-"reflection axioms":

$$
A \& A^\perp \vdash \neg A, \quad A \& A^\perp \vdash \neg A^\perp
$$

They are called $\&$-axioms in the literature. However, with respect to the semantic associated with quantum computing, each one is not an axiom on its own, in the sense that its truth value is not one. But the sum of the truth-values of the two axioms taken together is one.

Suppose now $A[\Delta \vdash (A^\perp[\Delta^\perp])$. By composition with the "axiom" $A \& A^\perp \vdash \neg A$ ($A \& A^\perp \vdash A^\perp$) we get the $\&$-explicit reflection rule:

$$
\& - \text{expl. refl} \quad A[\Delta \vdash \neg A \quad A \& A^\perp [\Delta^\perp]\vdash \neg A^\perp
$$

As we have completely solved the definitional equation (1) we can express quantum superposition in the object language with the composite proposition $A \& A^\perp$. Asserting: $A \& A^\perp$ (i.e. $\neg A \& \neg A^\perp$) is then equivalent to $(\neg A) \& (\neg A^\perp)$.

The logical expression of the qubit $|Q\rangle = a|0\rangle + b|1\rangle$ is then:
4. The logical connective @ for quantum entanglement

Two qubits $|Q\rangle_A = a|0\rangle_A + b|1\rangle_A$, $|Q\rangle_B = a'|0\rangle_B + b'|1\rangle_B$, are said entangled when the two qubits state $|Q\rangle_{AB}$ is not separable, i.e., $|Q\rangle_{AB} \neq |Q\rangle_A \otimes |Q\rangle_B$, where $\otimes$ is the tensor product in Hilbert spaces.

In particular, a two qubit state is maximally entangled when it is one of the four Bell states [12]:

$$|\Phi^\pm\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B \pm |1\rangle_A \otimes |1\rangle_B)$$

$$|\Psi^\pm\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B \pm |1\rangle_A \otimes |0\rangle_B).$$

For simplicity, in this paper we will consider only Bell states. As we have seen in Sect.3, expressing the qubit $|Q\rangle_A$ in logical terms leads to the compound proposition $Q_A \equiv A \& A^\perp$, where $\&$ stands for the connective "and".

In the same way, we can associate a proposition $B$ to the bit $B_1$ and its primitive negation $B^\perp$ to the bit $B_0$ so that the second qubit $|Q\rangle_B$ is expressed, in logical terms, by a second compound proposition $Q_B \equiv B \& B^\perp$. Bell states will be expressed, in logical terms, by the expression $Q_A \& Q_B$, where $\&$ is the new logical connective called "entanglement". Like all the other connectives, $\&$ will be defined by the reflection principle, which translates metalinguage into object language. We have at our disposal a meta-language which comes from our knowledge of the physical structure of Bell states. This leads us to figure out the logical structure for, say, the Bell states $|\Phi^\pm\rangle_{AB}$, namely $- (A \& \phi B) \& (A^\perp \& \phi B^\perp)$. Similarly, the logical structure for the Bell states $|\Psi^\pm\rangle_{AB}$ will be:

$- (A \& \phi B) \& (A^\perp \& \phi B^\perp)$. In the following, we will consider only the logical expression for the states $|\Phi^\pm\rangle_{AB}$, as the case for $|\Psi^\pm\rangle_{AB}$ is obtained exchanging $A$ with $A^\perp$. Eventually, we get the following definition: “Two compound propositions $Q_A \equiv A \& A^\perp$, $Q_B \equiv B \& B^\perp$ will be said (maximally) entangled if they are linked by the connective $\&$ = "entanglement". The definitional equation for $\&$ is:

$$\Gamma \vdash Q_A \& Q_B \quad \text{iff} \quad \Gamma \vdash A, B \quad \text{and} \quad \Gamma \vdash A^\perp, B^\perp$$

On the right-hand side of the definitional equation, we have the meta-language, coming from our knowledge of the physical structure of Bell states. On the left-hand side, instead, we have the object language. Also, it should be noticed that, on the right-hand side of the definitional equation, each of the two commas is reflected into a $\phi$ while the metalinguistic link and is reflected into $\&$ Thus the connective $\&$ is an additive as well as multiplicative connective (more exactly, an additive conjunction and a multiplicative disjunction) which reflects two kinds of "and" on the right: one outside the sequent (and) and one inside the sequent (the comma). Finally, the connective $\&$, is a derived connective which, nevertheless, has its own definitional equation: this is a new result in logic. Solving the definitional equation for $\&$ leads to the following rules:

@-formation
$$\Gamma \vdash A, B \quad \Gamma \vdash A^\perp, B^\perp \quad \Gamma \vdash Q_A \& Q_B$$

@-implicit reflection
$$\Gamma \vdash Q_A \& Q_B \quad \Gamma \vdash A, B$$

(i) $\Gamma \vdash Q_A \& Q_B \quad \Gamma \vdash A^\perp, B^\perp$

(ii) $\Gamma \vdash A, B \quad \Gamma \vdash A^\perp, B^\perp$

@-axioms
$$Q_A \& Q_B \vdash A, B \quad \text{(i)} \quad Q_A \& Q_B \vdash A^\perp, B^\perp \quad \text{(ii)}$$

@-explicit reflection
$$A \& Q_A \& Q_B \vdash A, B \quad \text{(i)} \quad A^\perp \& Q_A \& Q_B \vdash A^\perp, B^\perp \quad \text{(ii)}$$

Eq. (4) is equivalent to the @-definitional equation from the right hand side to the left hand side.
Eqs. (5) are equivalent to the $@$-definitional equation from the left hand side to the right hand side. The $@$-axioms in (6) are obtained from (5), by the trivialization procedure, that is, setting $\Gamma = Q_A @ Q_B$. The $@$-explicit reflection rules (i), (ii) in (7) are obtained by composition of the $@$-axioms (i) and (ii) in (6) with the premises $A \vdash \Delta$ and $B \vdash \Delta'$, and $A^\perp \vdash \Delta$ and $B^\perp \vdash \Delta'$, respectively.

The properties of $@$ are:

1) **Commutativity**:

$$Q_A @ Q_B \equiv Q_B @ Q_A$$

(8)

Commutativity of $@$ holds if and only if, the exchange rule is assumed (on the right). And in fact, exchange is a valid rule in Basic logic.

2) **Semi-distributivity**

From the definitional equation of $@$ with $\Gamma = \emptyset$, that is:

$$\neg Q_A @ Q_B \quad \text{iff} \quad \neg A, B \quad \text{and} \quad \neg A^\perp, B^\perp$$

we get:

$$(A & A^\perp) @ (B & B^\perp) \equiv (A \otimes B) \& (A^\perp \otimes B^\perp)$$

(9)

We see that two terms are missing in (9) namely $(A \otimes B^\perp)$ and $(A^\perp \otimes B)$, so that $@$ has distributivity with absorption, which we call semi-distributivity.

3) **Duality**

Let us define now the dual of $@$: $(Q_A @ Q_B)^\perp \equiv [A \otimes (B \& B^\perp)] = (A \otimes B) \| (A^\perp \otimes B^\perp)$

And let us call it $\&$, that is: $(Q_A @ Q_B)^\perp \equiv Q_A \& Q_B$ (vice-versa, the dual of $\&$ is $@$: $(Q_A \& Q_B)^\perp \equiv Q_A @ Q_B$).

The definition of the dual of $@$ is then:

$$Q_A @ Q_B \equiv (A \otimes B) \| (A^\perp \otimes B^\perp)$$

(10)

4) **Non-Associativity**:

$$Q_A @ (Q_B @ Q_C) \neq (Q_A @ Q_B) @ Q_C$$

(11)

To discuss associativity of $@$ a third qubit $Q_C$ is needed, and $Q_A @ (Q_B @ Q_C) \equiv (Q_A @ Q_B) @ Q_C$ cannot be demonstrated in Basic logic, as $Q_C$ acts like a context on the right.

5) **Non-idempotence**

$$Q_A @ Q_A \neq Q_A$$

(12)

We try proving $Q_A @ Q_A \neq Q_A$.

Let us try first $Q_A @ Q_A \vdash Q_A$.

There are no rules of Basic logic that we can use in the derivation, which can lead to a proof:

$$Q_A @ Q_A \vdash Q_A$$

And, as the cut-elimination theorem holds in Basic logic [14] because of visibility, we are sure that there are no other rules leading to a proof.

Let us try now the other way around: $Q_A \vdash Q_A @ Q_A$.

The only rule we can use in the derivation is the $@$-formation rule, and there are no further rules in Basic logic, which would lead to a proof:

$$\frac{Q_A \vdash Q_A, Q_A}{Q_A \vdash Q_A @ Q_A \quad \text{form.}}$$

And, again, because of cut-elimination, we are sure that there are no other rules leading to a proof.

For the sake of the physical interpretation, we show now that in the case the contraction and weakening rules did hold, the proof would be possible.
Let us prove first $Q_A @ Q_A \vdash Q_A$.

\[
\frac{Q_A \vdash Q_A}{Q_A \vdash Q_A @ Q_A} \quad \frac{Q_A @ Q_A \vdash Q_A}{Q_A @ Q_A \vdash Q_A} @ \text{- expl.refl.} \\
\frac{Q_A @ Q_A \vdash Q_A \cdot Q_A}{Q_A @ Q_A \vdash Q_A} \quad \text{contr.}
\]

Let us prove now the other way around $Q_A \vdash Q_A @ Q_A$.

\[
\frac{Q_A \vdash Q_A @ Q_A}{} \quad \frac{Q_A @ Q_A \vdash Q_A}{Q_A @ Q_A \vdash Q_A} \quad \text{weak.} \\
\frac{Q_A @ Q_A \vdash Q_A}{Q_A @ Q_A \vdash Q_A} \quad \text{- form.}
\]

It is impossible to prove $Q_A \vdash Q_A @ Q_A$ in Basic logic, because the weakening rule (in the step weak.) does not hold.

In conclusion, it is impossible to prove idempotence of $@$ in Basic logic, because of the absence of the two structural rules of weakening and contraction.

A less direct way to prove $Q_A @ Q_A \not= Q_A$ is the following. From the definition of $@$ in (9), by replacing $B$ with $A$ we get: $Q_A @ Q_A = (A \otimes A) \& (A \& A) = Q_A$, as $\phi$ is not idempotent. In fact, to demonstrate the idempotence of $\phi$ would require the validity of both the contraction and weakening rules. If one makes the formal proof one has to go both ways: to show that $A \vdash A \phi A$ does not hold because of the absence of the weakening rule, and that $A \phi A \vdash A$ does not hold because of the absence of the contraction rule.

If instead weakening and contraction did hold, then $\phi \equiv \lor (\otimes \equiv \&)$, and from the definition of $@$ in (9) we would get: $Q_A @ Q_A = (A \lor A) \& (A \& A) = Q_A$, because of the idempotence of $\lor$. In that case, from the definition of $\otimes$ in (10) we would also get $Q_A \otimes Q_A = (A \& A) \lor (A \lor A) = Q_A$, because of the idempotence of $\&$. Notice, in particular, that the formal proof that the dual of $\phi$, namely $\otimes =$"times" is not idempotent, would exchange the roles of the contraction and the weakening rules used in the proof done for $\phi$. More explicitly, to prove $A \vdash A \otimes A$ would require the contraction rule. But $A \vdash A \otimes A$ is just the logical interpretation of quantum cloning: $|\Psi\rangle \rightarrow |\Psi\rangle \otimes |\Psi\rangle$. The fact that $\otimes$ is not idempotent, leads to the result that the dual of $@$, namely $\otimes$, is not idempotent either. Then, $@ (\otimes)$ is not idempotent because $\phi (\otimes)$ is not idempotent. This illustrates the obvious physical situation: self-entanglement (entanglement of a qubit with itself) is impossible as it would require a quantum clone, which is forbidden by the no-cloning theorem. In a sense, one can say that the two main no-go theorems of quantum computing, namely the no-cloning and no-erase theorems are (logically) dual to each other, and the no self-entanglement "corollary" is a consequence of the first one, when entanglement is expressed in terms of $\otimes$, and a consequence of the second one, when entanglement is expressed by the dual, $@$.

On the other hand, it turns out that the meaning of "no self-entanglement" is much more profound in logic. In fact, affirming that in a certain formal language it is impossible to get a (compound) proposition (maximally) entangled with itself means that the language under study does not lead to self-referential sentences in the meta-language.

Schematically:

\[
\begin{array}{cccc}
\text{No contraction} & \text{No weakening} & \text{BASIC LOGIC} & \text{No contraction} & \text{No weakening} \\
\downarrow & \downarrow & \text{Symmetry} & \downarrow & \downarrow \\
A \vdash A \otimes A & \text{Cannot be proved} & A \otimes A \rightarrow \phi & A \phi A \vdash A & \text{Cannot be proved} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
Q_A \otimes Q_A \not= Q_A & \text{No-idempotence of} \@ & Q_A @ Q_A \not= Q_A & Q_A @ Q_A \not= Q_A
\end{array}
\]
QUANTUM COMPUTING

No-cloning

\[ |\Psi\rangle \neq |\Psi\rangle \otimes |\Psi\rangle \]

No-erase

\[ |\Psi\rangle \otimes |\Psi\rangle \neq |\Psi\rangle \]

No self-entanglement

Notice that the property of the non-idempotence of @ (and of its dual §) deals with the object language. However, when this property is translated into a meta-language like natural language, we get:

\[ Q_A @ Q_A \neq Q_A \]

This just expresses the impossibility of having self-referential sentences. It is not a paradox. However, many classical “paradoxes” like the Liar paradox: “This sentence is false” look very much alike the property of the non-idempotence of @, which in fact is a theorem. The reason is that in our classical reasoning, the concept of entanglement is missing, and moreover, when we try to formalize the “paradox” in any other logic (but basic logic) which has not the connective @, we fail. See Fig.2.

Fig.2
The Liar “paradox” revisited

5. The EPR rule
It can be shown [11] that the cut:

\[
\frac{\neg Q_A}{\neg A} A \quad \frac{Q_A}{\neg A} \quad cut 
\]

(13)
corresponds, in physical terms, to measure the qubit \( |Q\rangle_A \) in state \( |1\rangle_A \) (with probability \( |\beta|^2 \)). In the same way, the cut:

\[
\frac{\neg A^\perp}{\neg Q_A} A^\perp \quad A^\perp \quad cut 
\]
corresponds to measure the qubit \( |Q\rangle_A \) in state \( |0\rangle_A \) (with probability \( |\epsilon|^2 \)).

The cut (over entanglement) is:
Performing the cut in (14) corresponds, in physical terms, to measure the state $\left| 1 \rightangle_A \left| 1 \rightangle_B$. If we replace $A$ and $B$ in (14) with $A^\perp, B^\perp$ the cut corresponds to measure the state $\left| 0 \rightangle_A \left| 0 \rightangle_B$. It should be noticed, that, if we make a measurement of $Q_A$ (supposed entangled with $Q_B$) and get $A$, then by semi-distributivity of $\otimes$, we have:

$$A \otimes Q_B \equiv A \otimes (B \otimes B^\perp) \equiv A \otimes B$$

As it is well known, if two quantum systems $S_A$ and $S_B$ are entangled, they share a unique quantum state, and even if they are far apart, a measurement performed on $S_A$ influences any subsequent measurement performed on $S_B$ (the EPR “paradox” [6]). Let us consider Alice, who is an observer for system $S_A$, which is the qubit $Q_A$, that is, she can perform a measurement of $Q_A$. There are two possible outcomes, with equal probability 1/2:

i) Alice measures 1, and the Bell state collapses to $\left| 1 \rightangle_A \left| 1 \rightangle_B$.

ii) Alice measures 0, and the Bell state collapses to $\left| 0 \rightangle_A \left| 0 \rightangle_B$.

Now, let us suppose Bob is an observer for system $S_B$ (the qubit $Q_B$). If Alice has measured 1, any subsequent measurement of $Q_B$ performed by Bob always returns 1. If Alice measured 0, instead, any subsequent measurement of $Q_B$ performed by Bob always returns 0. To discuss the EPR paradox in logical terms, we introduce the EPR rule:

$$\Gamma \vdash -A, B \quad \text{\@ - impl. refl.}$$

Where the semi-distributivity of $\otimes$, i.e. $A \otimes Q_B \equiv A, B$ has been used in the step “@-impl refl.”. Notice that the consequences of the EPR rule are the same of the cut over entanglement (14), because of semi-distributivity of $\otimes$. It was believed that no other rule existed, a part from the cut rule, or at least some rule equivalent to it, which could cut a formula in a logical derivation. Nevertheless, the EPR rule does cut a formula, but it can be proved that it is not equivalent to the cut rule over entanglement (and, vice-versa, the cut rule over entanglement is not equivalent to the EPR rule). This is a new result in logic.

Let us show first that the EPR rule is not equivalent to the cut rule. We start with the premises of the EPR rule and apply the cut rule:

$$\Gamma \vdash Q_A \otimes Q_B \quad Q_A \otimes Q_B \vdash -A, B \quad \text{\@ - axiom}$$

$$\text{cut}$$

It is clear that it is impossible to demonstrate that the EPR rule is equivalent to the cut rule (over entanglement) in Basic logic, where we don't have the structural rules of weakening and contraction. And in any logic with structural rules, the connective entanglement disappears, and the EPR rule collapses to the cut rule.

Now, we will show the vice-versa, i.e., that the cut rule (over entanglement) is not equivalent to the EPR rule. We start with the premises of the cut rule (14) and apply the cut rule (16):

$$\Gamma \vdash -A, B, Q_A \quad Q_A \vdash -A \quad \text{\textit{cut}}$$

$$\Gamma \vdash -A, B$$

$$\text{weak.L}$$

$$\Gamma \vdash -A, B, Q_A \quad Q_A \vdash -A \quad \text{\textit{cut}}$$

$$\Gamma \vdash -A, B$$

$$\text{contr.R}$$
Where, in (18), $EPR^C$ means EPR rule in presence of contexts (here $Q_A @ Q_B$ on the left and $B$ on the right). But contexts are absent in Basic logic (visibility). Furthermore, the weakening rule is not present in Basic logic. These facts lead to the conclusion that in Basic logic it is impossible to prove that the cut is equivalent to the EPR rule. Moreover, this is impossible also in sub-structural rules with contexts (like BL, BR, and BLR (linear logic) because one cannot use weakening, and in structural logics because the connective entanglement disappears.

The EPR rule is a new kind of meta-rule peculiar of entanglement, which is possible only in Basic logic. It is a stronger rule (although less general) than the cut, as it uses a weaker premise to yield the same result. Hence, instead of proving $Q_A @ Q_B$ in (14) that is $Q_A @ Q_B$, we can just prove $Q_A$, i.e., $Q_A$, perform the usual cut (13) (over $Q_A$), and leave the result $A$ entangled with $Q_B$. Roughly speaking, if two compound propositions are (maximally) entangled, it is sufficient to prove only one of them. This is the logical analogue of the EPR "paradox". Now, let us consider the two EPRs:

\[
\Gamma \vdash Q_A @ Q_B \quad \frac{Q_A @ Q_B \vdash A, B}{Q_A @ Q_B, Q_A \vdash A, B} \text{weak.L} \quad \frac{Q_A @ Q_B, Q_A \vdash A, B}{\Gamma, Q_A @ Q_B \vdash A @ Q_B, B} \quad EPR^C
\]

(18)

6. Conclusions

Basic logic, once endowed with the new connective “entanglement” and the EPR rule, provides the unique adequate sequent calculus for quantum computing. We list below the main features of quantum information and quantum computing, and the corresponding required properties for the associated logic.

The main features of quantum computing are:
1) Quantum Information cannot be copied (no-cloning theorem).
2) Quantum Information cannot be deleted (no-erase theorem).
3) Heisenberg uncertainty principle.
4) Quantum superposition
5) Quantum entanglement
6) Quantum non-locality, EPR “paradox”.
7) Irreversibility of quantum measurement.

The corresponding logical requirements are:
1') No contraction rule
2') No weakening rule
3') Non-distributivity, then no free contexts on both sides.
4') Connective $&=$ "superposition"
5') Connective $@ =$ "entanglement"
6') The EPR rule
7') The Cut rule

Requirements 1’-3’ exclude all logics apart from Basic logic B (and BR, BL, for more than two qubits).
B satisfies the remaining requirements 4’-7’.

Among others, two hints for possible physical implications are:
i) The relation between the logical B-triangle of vertices B, BR, BL, and ’t Hooft’s quantum determinism at Planck scale – physics [17] [18] when a notion of duality is included.
ii) The relation between the B-triangle and Vitiello’s double universe [19]. See fig.3
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References